# Inflation and Correlation in Claims Run-Off Triangles

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# Objective

- modeling dependence **within** claims reserving triangles
- modeling dependence **between** claims reserving triangles

## Joint work with

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▷ Michael Merz (University of Hamburg)

▷ Robert Salzmann (RiskLab, ETH Zurich)

# Claims development triangle

| accident year $i$ | development year $j$ | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | | to be **predicted** | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

- $X_{i,j}$ denotes the payments for accident year $i$ in development year $j$.
- Thus, $X_{i,j}$ is paid in calendar year $k = i + j$.

# Observations at time $I$

| accident year $i$ | development year $j$: 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | | to be **predicted** | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

- Observed payments at time $I = 2010$ (upper triangle)

$$\mathcal{D}_I = \{X_{i,j};\ i + j \leq I\}.$$

# Claims prediction and claims reserves

| accident year $i$ | development year $j$: 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | to be **predicted** | | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

- Predict payments $\mathcal{D}_I^c = \{X_{i,j};\ i+j > I\}$ (lower triangle).
- **Best-estimate reserves**: $\mathcal{R}_I = \sum_{i+j>I} \mathbb{E}\left[X_{i,j} \middle| \mathcal{D}_I\right]$.

# Independence between accident years $i$

| accident year $i$ | development year $j$: 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | | | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

- All classical stochastic claims reserving methods assume **independence** between payments $X_{i,j}$ of different accident years $i$.

# Claims inflation and calendar year effects

| accident year $i$ | development year $j$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | | | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

- Payments in different accident years $i$ are **not independent**.

- Different portfolios $X_{i,j,n}$ for $n = 1, \ldots, N \Rightarrow$ **multivariate model**.

# Claims inflation and calendar year effects

| accident year $i$ | development year $j$: 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1994 | 13'109 | 7'246 | 982 | 706 | 358 | 257 | 339 | 161 | 334 | 172 | 35 | 205 | 56 | 32 | 2 | 7 | 1 |
| 1995 | 14'457 | 7'581 | 589 | 487 | 124 | 74 | 128 | 50 | 474 | 12 | 72 | 63 | 141 | 286 | 2 | 10 | |
| 1996 | 16'075 | 6'597 | 1'081 | 299 | 154 | 551 | 29 | 21 | 16 | 65 | 98 | 415 | 280 | 24 | 27 | | |
| 1997 | 15'682 | 7'782 | 1'001 | 587 | 477 | 179 | 44 | 18 | 65 | 240 | 7 | 64 | 4 | 17 | | | |
| 1998 | 16'551 | 7'155 | 921 | 946 | 473 | 69 | 168 | 198 | 220 | 17 | 6 | 4 | 7 | | | | |
| 1999 | 15'439 | 8'357 | 1'070 | 451 | 822 | 15 | 21 | 30 | 559 | 54 | 18 | 123 | | | | | |
| 2000 | 14'629 | 7'016 | 1'181 | 773 | 1'393 | 442 | 42 | 73 | 55 | 105 | 14 | | | | | | |
| 2001 | 17'585 | 8'703 | 1'335 | 316 | 396 | 303 | 77 | 44 | 766 | 777 | | | | | | | |
| 2002 | 17'419 | 8'522 | 1'125 | 695 | 282 | 434 | 244 | 157 | 70 | | | | | | | | |
| 2003 | 16'665 | 8'705 | 1'539 | 702 | 118 | 132 | 1'969 | 14 | | | | | | | | | |
| 2004 | 15'471 | 8'274 | 1'372 | 1'261 | 593 | 425 | 84 | | | | | | | | | | |
| 2005 | 15'103 | 8'290 | 3'416 | 882 | 370 | 1'122 | | | | | | | | | | | |
| 2006 | 14'540 | 8'102 | 929 | 556 | 83 | | | | | | | | | | | | |
| 2007 | 14'590 | 7'746 | 1'104 | 589 | | | | | | | | | | | | | |
| 2008 | 13'967 | 7'548 | 1'088 | | | | | | | | | | | | | | |
| 2009 | 12'930 | 7'181 | | | | | | | | | | | | | | | |
| 2010 | 12'539 | | | | | | | | | | | | | | | | |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1997 | 2'998 | 5'159 | 4'568 | 4'751 | 4'187 | 2'126 | 1'894 | 332 | 383 | 553 | 87 | 179 | 341 | 95 |
| 1998 | 2'484 | 5'183 | 5'980 | 5'521 | 3'772 | 2'228 | 1'463 | 941 | 218 | 44 | 209 | 136 | 39 | |
| 1999 | 2'596 | 5'907 | 7'458 | 5'291 | 3'949 | 2'039 | 735 | 898 | 546 | 132 | 516 | 71 | | |
| 2000 | 4'247 | 9'412 | 6'707 | 6'999 | 3'723 | 3'270 | 2'458 | 1'057 | 30 | 1'202 | 127 | | | |
| 2001 | 4'946 | 8'970 | 8'510 | 5'658 | 3'982 | 4'018 | 1'021 | 516 | 810 | 13 | | | | |
| 2002 | 3'585 | 8'694 | 8'565 | 7'204 | 4'685 | 3'608 | 2'077 | 1'262 | 875 | | | | | |
| 2003 | 4'618 | 9'658 | 9'035 | 7'691 | 6'060 | 4'338 | 2'277 | 1'160 | | | | | | |
| 2004 | 4'787 | 10'871 | 12'028 | 10'114 | 5'094 | 5'248 | 2'971 | | | | | | | |
| 2005 | 4'876 | 12'285 | 11'611 | 9'679 | 8'254 | 4'260 | | | | | | | | |
| 2006 | 8'684 | 14'287 | 13'146 | 11'650 | 9'321 | | | | | | | | | |
| 2007 | 6'991 | 14'857 | 18'648 | 18'214 | | | | | | | | | | |
| 2008 | 7'748 | 18'656 | 25'230 | | | | | | | | | | | |
| 2009 | 9'806 | 28'842 | | | | | | | | | | | | |
| 2010 | 10'216 | | | | | | | | | | | | | |

- Payments in different accident years $i$ are **not independent**.
- Different portfolios $X_{i,j,n}$ for $n = 1, \ldots, N \Rightarrow$ **multivariate model**.

# Chain-ladder model

Define cumulative payments of portfolio $n = 1, \ldots, N$ by

$$C_{i,j,n} \;=\; X_{i,0,n} + \ldots + X_{i,j,n} \;=\; \sum_{l=0}^{j} X_{i,l,n}.$$

Chain-ladder models are of multiplicative type

$$C_{i,j,n} = C_{i,j-1,n} \;\exp\left\{\xi_{i,j,n}\right\},$$

with individual log-link ratios

$$\xi_{i,j,n} = \log\left(\frac{C_{i,j,n}}{C_{i,j-1,n}}\right).$$

# Definition of the multivariate model

For $i \in \{1, \ldots, I\}$ and $j \in \{0, \ldots, J\}$ we define the random vectors

$$\xi_{i,j} = (\xi_{i,j,1}, \ldots, \xi_{i,j,N})' \in \mathbb{R}^N \qquad \text{(portfolios } n\text{)},$$

$$\boldsymbol{\xi}_i = (\xi'_{i,0}, \ldots, \xi'_{i,J})' \in \mathbb{R}^a \qquad \text{(development years } j\text{)},$$

$$\boldsymbol{\xi} = (\boldsymbol{\xi}'_1, \ldots, \boldsymbol{\xi}'_I)' \in \mathbb{R}^d \qquad \text{(accident years } i\text{)},$$

with $a = N(J+1)$ and $d = aI$.

Recall: $\quad C_{i,j,n} = C_{i,j-1,n} \exp\{\xi_{i,j,n}\}$.

$\Longrightarrow$ Random vector $\boldsymbol{\xi}$ contains *all* individual log-link ratios.

# Bayesian **multivariate chain-ladder model**

## Model assumptions.

- Conditionally, given $\boldsymbol{\Theta} \in \mathbb{R}^a$,
  - ▷ $\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}}$ is **multivariate** Gauss with covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$,
  - ▷ and mean $\mathbb{E}\left[\boldsymbol{\xi}_i \middle| \boldsymbol{\Theta}\right] = \boldsymbol{\Theta}$ for **all** $i \in \{1, \ldots, I\}$.
- The parameter has multivariate **prior** distribution $\boldsymbol{\Theta} \overset{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}, T)$ .

# First consequences of the model assumptions

- The mean $\mathbb{E}[\boldsymbol{\xi}_i | \boldsymbol{\Theta}] = \boldsymbol{\Theta}$ does **not** depend on $i \in \{1, \ldots, I\}$:

$$\mathbb{E}[\xi_{i,j,n} | \boldsymbol{\Theta}] = \Theta_{j,n},$$

with $\Theta_{j,n}$ reflecting the chain-ladder factor of portfolio $n$ for development year $j$.

- Covariance matrix $\Sigma$ allows for **any correlation structure** between (all) individual log-link ratios in $\boldsymbol{\xi}$, for example, for $(i, j, n) \neq (l, k, m)$

$$\mathrm{Corr}(\xi_{i,j,n}, \xi_{l,k,m} | \boldsymbol{\Theta}) = \rho > 0 \qquad \text{for all } i + j = l + k.$$

These are exactly claims inflation and calendar year effects.

# Bayesian multivariate chain-ladder model

## Model assumptions.

- Conditionally, given $\boldsymbol{\Theta} \in \mathbb{R}^a$,
  - ▷ $\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}}$ is **multivariate** Gauss with covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$,
  - ▷ and mean $\mathbb{E}\left[\boldsymbol{\xi}_i \middle| \boldsymbol{\Theta}\right] = \boldsymbol{\Theta}$ for **all** $i \in \{1, \ldots, I\}$.
- The parameter has multivariate **prior** distribution $\boldsymbol{\Theta} \overset{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}, T)$ .

# Consequences of the model assumptions

- Appropriate choices of $\Sigma$ allow to model **any correlation structure**, such as inflation, within the triangles and between the triangles:

$$\boldsymbol{\xi}|_{\{\boldsymbol{\Theta}\}} \stackrel{(\mathrm{d})}{\sim} \mathcal{N}(A\boldsymbol{\Theta}, \Sigma),$$

for an appropriate matrix $A = (\mathbb{1}, \ldots, \mathbb{1})' \in \mathbb{R}^{d \times a}$ (different accident years $i$ are characterized by the same $\Theta_{j,n}$).

- The choice of the prior distribution allows to implement **parameter uncertainty** and **expert knowledge** about chain-ladder factors

$$\boldsymbol{\Theta} \stackrel{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}, T).$$

# Predictive distribution in lower triangles

Denote by $\boldsymbol{\xi}^{\mathcal{D}_I}$ the observations in the upper triangles $\mathcal{D}_I$ and by $\boldsymbol{\xi}^{\mathcal{D}_I^c}$ the unobserved components in the lower triangles $\mathcal{D}_I^c$.

**Theorem 1 (predictive distribution).**

We have posterior distribution

$$\boldsymbol{\xi}^{\mathcal{D}_I^c}\big|_{\left\{\boldsymbol{\xi}^{\mathcal{D}_I}\right\}} \stackrel{(\mathrm{d})}{\sim} \mathcal{N}(\boldsymbol{\mu}^{\mathrm{post}}, S^{\mathrm{post}}),$$

with explicit (credibility) formulas for $\boldsymbol{\mu}^{\mathrm{post}}$ and $S^{\mathrm{post}}$.

$S^{\mathrm{post}}$ contains both covariance matrix $\Sigma$ *and* parameter uncertainty $T$.

# Claims reserves and prediction uncertainty

## Corollary 2.

The ultimate claim predictor is given by

$$\widehat{C}_{i,J,n} = \mathbb{E}\left[C_{i,J,n} \middle| \mathcal{D}_I\right] = C_{i,I-i,n} \; \exp\left\{\mathbf{e}'_{i,n} \; \boldsymbol{\mu}^{\text{post}} + \frac{1}{2}\, \mathbf{e}'_{i,n} \; S^{\text{post}} \; \mathbf{e}_{i,n}\right\},$$

for appropriate projections $\mathbf{e}_{i,n}$.

- There are similar closed formula for prediction uncertainty (MSEP).
- Closed formula for the one-year claims development result (CDR) uncertainty is also available.

# Summary

- We can choose

  ⋆ **any correlation structure** $\Sigma$ between individual log-link ratios $\xi_{i,j,n}$'s, and
  ⋆ **any prior information** $\boldsymbol{\mu}$ and $T$ on the parameter space $\boldsymbol{\Theta}$,

  and then Theorem 1 provides the **posterior distribution** of $\boldsymbol{\xi}^{\mathcal{D}_I^c}$, given the observations $\boldsymbol{\xi}^{\mathcal{D}_I}$ in *closed form*.

- We have closed form solutions for

  ⋆ the ultimate claim predictor and the claims reserves (Corollary 2),
  ⋆ MSEP for total uncertainty and one-year CDR uncertainty.

- Our model allows in **solvency models** for a

  **bottom-up calibration of correlation**.

# Case study from Merz-W.-Hashorva [1]

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59'966 | 103'186 | 91'360 | 95'012 | 83'741 | 42'513 | 37'882 | 6'649 | 7'669 | 11'061 | -1'738 | 3'572 | 6'823 | 1'893 |
| 2 | 49'685 | 103'659 | 119'592 | 110'413 | 75'442 | 44'567 | 29'257 | 18'822 | 4'355 | 879 | 4'173 | 2'727 | -776 | |
| 3 | 51'914 | 118'134 | 149'156 | 105'825 | 78'970 | 40'770 | 14'706 | 17'950 | 10'917 | 2'643 | 10'311 | 1'414 | | |
| 4 | 84'937 | 188'246 | 134'135 | 139'970 | 74'450 | 65'401 | 49'165 | 21'136 | 596 | 24'048 | 2'548 | | | |
| 5 | 98'921 | 179'408 | 170'201 | 113'161 | 79'641 | 80'364 | 20'414 | 10'324 | 16'204 | -265 | | | | |
| 6 | 71'708 | 173'879 | 171'295 | 144'076 | 93'694 | 72'161 | 41'545 | 25'245 | 17'497 | | | | | |
| 7 | 92'350 | 193'157 | 180'707 | 153'816 | 121'196 | 86'753 | 45'547 | 23'202 | | | | | | |
| 8 | 95'731 | 217'413 | 240'558 | 202'276 | 101'881 | 104'966 | 59'416 | | | | | | | |
| 9 | 97'518 | 245'700 | 232'223 | 193'576 | 165'086 | 85'200 | | | | | | | | |
| 10 | 173'686 | 285'730 | 262'920 | 232'999 | 186'415 | | | | | | | | | |
| 11 | 139'821 | 297'137 | 372'968 | 364'270 | | | | | | | | | | |
| 12 | 154'965 | 373'115 | 504'604 | | | | | | | | | | | |
| 13 | 196'124 | 576'847 | | | | | | | | | | | | |
| 14 | 204'325 | | | | | | | | | | | | | |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 114'423 | 133'538 | 65'021 | 31'358 | 27'139 | -377 | 9'889 | 4'477 | -316 | 7'108 | -1'035 | 103 | 209 | -109 |
| 2 | 152'296 | 152'879 | 71'438 | 41'686 | 22'009 | 25'315 | 7'961 | 4'843 | -113 | 1'593 | 848 | 4'383 | -1'164 | |
| 3 | 144'325 | 162'919 | 106'365 | 50'432 | 55'224 | 7'951 | 8'234 | 1'409 | 2'061 | 669 | 176 | 977 | | |
| 4 | 145'904 | 161'732 | 79'458 | 46'642 | 29'384 | 15'811 | 3'598 | 5'527 | -2'484 | 462 | -1'018 | | | |
| 5 | 170'333 | 171'168 | 92'601 | 36'227 | 11'872 | 18'760 | 3'180 | 3'538 | 948 | -875 | | | | |
| 6 | 189'643 | 171'480 | 85'734 | 61'226 | 18'479 | 13'556 | 7'523 | 1'964 | 88 | | | | | |
| 7 | 179'022 | 217'202 | 101'080 | 56'183 | 28'362 | 29'791 | 11'244 | 12'568 | | | | | | |
| 8 | 205'908 | 210'139 | 104'397 | 45'277 | 34'888 | 30'193 | 17'563 | | | | | | | |
| 9 | 210'951 | 215'478 | 98'618 | 62'846 | 52'435 | 22'824 | | | | | | | | |
| 10 | 213'426 | 295'796 | 140'211 | 82'259 | 59'209 | | | | | | | | | |
| 11 | 249'508 | 330'502 | 142'126 | 122'023 | | | | | | | | | | |
| 12 | 258'425 | 427'587 | 229'097 | | | | | | | | | | | |
| 13 | 368'762 | 540'304 | | | | | | | | | | | | |
| 14 | 394'997 | | | | | | | | | | | | | |

# Case study from Merz-W.-Hashorva [1]

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59'966 | 103'186 | 91'360 | 95'012 | 83'741 | 42'513 | 37'882 | 6'649 | 7'669 | 11'061 | -1'738 | 3'572 | 6'823 | 1'893 |
| 2 | 49'685 | 103'659 | 119'592 | 110'413 | 75'442 | 44'567 | 29'257 | 18'822 | 4'355 | 879 | 4'173 | 2'727 | -776 | |
| 3 | 51'914 | 118'134 | 149'156 | 105'825 | 78'970 | 40'770 | 14'706 | 17'950 | 10'917 | 2'643 | 10'311 | 1'414 | | |
| 4 | 84'937 | 188'246 | 134'135 | 139'970 | 74'450 | 65'401 | 49'165 | 21'136 | 596 | 24'048 | 2'548 | | | |
| 5 | 98'921 | 179'408 | 170'201 | 113'161 | 79'641 | 80'364 | 20'414 | 10'324 | 16'204 | -265 | | | | |
| 6 | 71'708 | 173'879 | 171'295 | 144'076 | 93'694 | 72'161 | 41'545 | 25'245 | 17'497 | | | | | |
| 7 | 92'350 | 193'157 | 180'707 | 153'816 | 121'196 | 86'753 | 45'547 | 23'202 | | | | | | |
| 8 | 95'731 | 217'413 | 240'558 | 202'276 | 101'881 | 104'966 | 59'416 | | | | | | | |
| 9 | 97'518 | 245'700 | 232'223 | 193'576 | 165'086 | 85'200 | | | | | | | | |
| 10 | 173'686 | 285'730 | 262'920 | 232'999 | 186'415 | | | | | | | | | |
| 11 | 139'821 | 297'137 | 372'968 | 364'270 | | | | | | | | | | |
| 12 | 154'965 | 373'115 | 504'604 | | | | | | | | | | | |
| 13 | 196'124 | 576'847 | | | | | | | | | | | | |
| 14 | 204'325 | | | | | | | | | | | | | |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 114'423 | 133'538 | 65'021 | 31'358 | 27'139 | -377 | 9'889 | 4'477 | -316 | 7'108 | -1'035 | 103 | 209 | -109 |
| 2 | 152'296 | 152'879 | 71'438 | 41'686 | 22'009 | 25'315 | 7'961 | 4'843 | -113 | 1'593 | 848 | 4'383 | -1'164 | |
| 3 | 144'325 | 162'919 | 106'365 | 50'432 | 55'224 | 7'951 | 8'234 | 1'409 | 2'061 | 669 | 176 | 977 | | |
| 4 | 145'904 | 161'732 | 79'458 | 46'642 | 29'384 | 15'811 | 3'598 | 5'527 | -2'484 | 462 | -1'018 | | | |
| 5 | 170'333 | 171'168 | 92'601 | 36'227 | 11'872 | 18'760 | 3'180 | 3'538 | 948 | -875 | | | | |
| 6 | 189'643 | 171'480 | 85'734 | 61'226 | 18'479 | 13'556 | 7'523 | 1'964 | 88 | | | | | |
| 7 | 179'022 | 217'202 | 101'080 | 56'183 | 28'362 | 29'791 | 11'244 | 12'568 | | | | | | |
| 8 | 205'908 | 210'139 | 104'397 | 45'277 | 34'888 | 30'193 | 17'563 | | | | | | | |
| 9 | 210'951 | 215'478 | 98'618 | 62'846 | 52'435 | 22'824 | | | | | | | | |
| 10 | 213'426 | 295'796 | 140'211 | 82'259 | 59'209 | | | | | | | | | |
| 11 | 249'508 | 330'502 | 142'126 | 122'023 | | | | | | | | | | |
| 12 | 258'425 | 427'587 | 229'097 | | | | | | | | | | | |
| 13 | 368'762 | 540'304 | | | | | | | | | | | | |
| 14 | 394'997 | | | | | | | | | | | | | |


| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59'966 | 103'186 | 91'360 | 95'012 | 83'741 | 42'513 | 37'882 | 6'649 | 7'669 | 11'061 | -1'738 | 3'572 | 6'823 | 1'893 |
| 2 | 49'685 | 103'659 | 119'592 | 110'413 | 75'442 | 44'567 | 29'257 | 18'822 | 4'355 | 879 | 4'173 | 2'727 | -776 | |
| 3 | 51'914 | 118'134 | 149'156 | 105'825 | 78'970 | 40'770 | 14'706 | 17'950 | 10'917 | 2'643 | 10'311 | 1'414 | | |
| 4 | 84'937 | 188'246 | 134'135 | 139'970 | 74'450 | 65'401 | 49'165 | 21'136 | 596 | 24'048 | 2'548 | | | |
| 5 | 98'921 | 179'408 | 170'201 | 113'161 | 79'641 | 80'364 | 20'414 | 10'324 | 16'204 | -265 | | | | |
| 6 | 71'708 | 173'879 | 171'295 | 144'076 | 93'694 | 72'161 | 41'545 | 25'245 | 17'497 | | | | | |
| 7 | 92'350 | 193'157 | 180'707 | 153'816 | 121'196 | 86'753 | 45'547 | 23'202 | | | | | | |
| 8 | 95'731 | 217'413 | 240'558 | 202'276 | 101'881 | 104'966 | 59'416 | | | | | | | |
| 9 | 97'518 | 245'700 | 232'223 | 193'576 | 165'086 | 85'200 | | | | | | | | |
| 10 | 173'686 | 285'730 | 262'920 | 232'999 | 186'415 | | | | | | | | | |
| 11 | 139'821 | 297'137 | 372'968 | 364'270 | | | | | | | | | | |
| 12 | 154'965 | 373'115 | 504'604 | | | | | | | | | | | |
| 13 | 196'124 | 576'847 | | | | | | | | | | | | |
| 14 | 204'325 | | | | | | | | | | | | | |

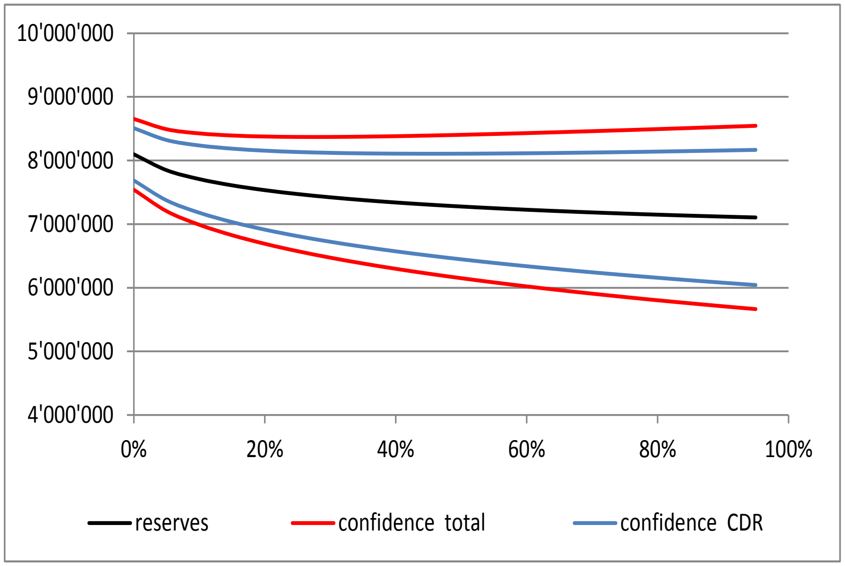
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 114'423 | 133'538 | 65'021 | 31'358 | 27'139 | -377 | 9'889 | 4'477 | -316 | 7'108 | -1'035 | 103 | 209 | -109 |
| 2 | 152'296 | 152'879 | 71'438 | 41'686 | 22'009 | 25'315 | 7'961 | 4'843 | -113 | 1'593 | 848 | 4'383 | -1'164 | |
| 3 | 144'325 | 162'919 | 106'365 | 50'432 | 55'224 | 7'951 | 8'234 | 1'409 | 2'061 | 669 | 176 | 977 | | |
| 4 | 145'904 | 161'732 | 79'458 | 46'642 | 29'384 | 15'811 | 3'598 | 5'527 | -2'484 | 462 | -1'018 | | | |
| 5 | 170'333 | 171'168 | 92'601 | 36'227 | 11'872 | 18'760 | 3'180 | 3'538 | 948 | -875 | | | | |
| 6 | 189'643 | 171'480 | 85'734 | 61'226 | 18'479 | 13'556 | 7'523 | 1'964 | 88 | | | | | |
| 7 | 179'022 | 217'202 | 101'080 | 56'183 | 28'362 | 29'791 | 11'244 | 12'568 | | | | | | |
| 8 | 205'908 | 210'139 | 104'397 | 45'277 | 34'888 | 30'193 | 17'563 | | | | | | | |
| 9 | 210'951 | 215'478 | 98'618 | 62'846 | 52'435 | 22'824 | | | | | | | | |
| 10 | 213'426 | 295'796 | 140'211 | 82'259 | 59'209 | | | | | | | | | |
| 11 | 249'508 | 330'502 | 142'126 | 122'023 | | | | | | | | | | |
| 12 | 258'425 | 427'587 | 229'097 | | | | | | | | | | | |
| 13 | 368'762 | 540'304 | | | | | | | | | | | | |
| 14 | 394'997 | | | | | | | | | | | | | |

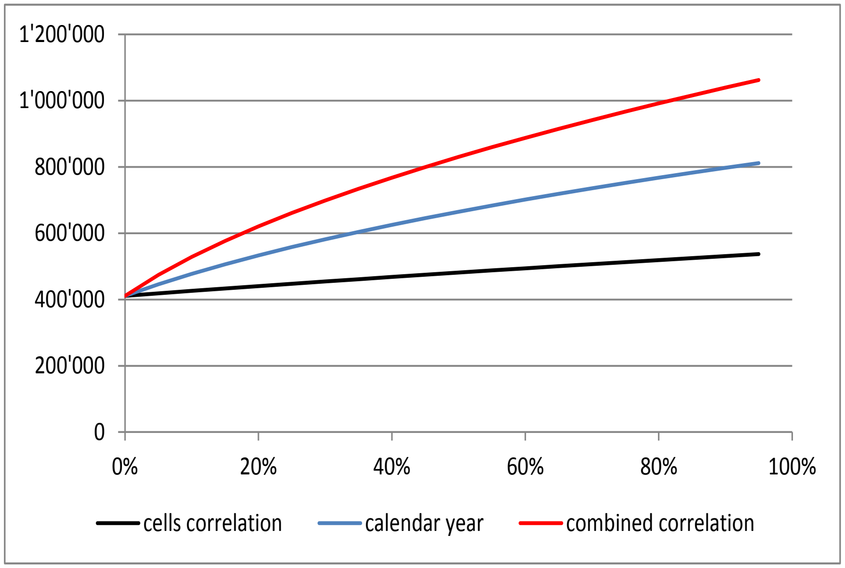

(1) point-wise cells correlation between triangles
(2) calendar year correlation within triangles
(3) combined: calendar year correlation within and between triangles


| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59'966 | 103'186 | 91'360 | 95'012 | 83'741 | 42'513 | 37'882 | 6'649 | 7'669 | 11'061 | -1'738 | 3'572 | 6'823 | 1'893 |
| 2 | 49'685 | 103'659 | 119'592 | 110'413 | 75'442 | 44'567 | 29'257 | 18'822 | 4'355 | 879 | 4'173 | 2'727 | -776 | |
| 3 | 51'914 | 118'134 | 149'156 | 105'825 | 78'970 | 40'770 | 14'706 | 17'950 | 10'917 | 2'643 | 10'311 | 1'414 | | |
| 4 | 84'937 | 188'246 | 134'135 | 139'970 | 74'450 | 65'401 | 49'165 | 21'136 | 596 | 24'048 | 2'548 | | | |
| 5 | 98'921 | 179'408 | 170'201 | 113'161 | 79'641 | 80'364 | 20'414 | 10'324 | 16'204 | -265 | | | | |
| 6 | 71'708 | 173'879 | 171'295 | 144'076 | 93'694 | 72'161 | 41'545 | 25'245 | 17'497 | | | | | |
| 7 | 92'350 | 193'157 | 180'707 | 153'816 | 121'196 | 86'753 | 45'547 | 23'202 | | | | | | |
| 8 | 95'731 | 217'413 | 240'558 | 202'276 | 101'881 | 104'966 | 59'416 | | | | | | | |
| 9 | 97'518 | 245'700 | 232'223 | 193'576 | 165'086 | 85'200 | | | | | | | | |
| 10 | 173'686 | 285'730 | 262'920 | 232'999 | 186'415 | | | | | | | | | |
| 11 | 139'821 | 297'137 | 372'968 | 364'270 | | | | | | | | | | |
| 12 | 154'965 | 373'115 | 504'604 | | | | | | | | | | | |
| 13 | 196'124 | 576'847 | | | | | | | | | | | | |
| 14 | 204'325 | | | | | | | | | | | | | |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 114'423 | 133'538 | 65'021 | 31'358 | 27'139 | -377 | 9'889 | 4'477 | -316 | 7'108 | -1'035 | 103 | 209 | -109 |
| 2 | 152'296 | 152'879 | 71'438 | 41'686 | 22'009 | 25'315 | 7'961 | 4'843 | -113 | 1'593 | 848 | 4'383 | -1'164 | |
| 3 | 144'325 | 162'919 | 106'365 | 50'432 | 55'224 | 7'951 | 8'234 | 1'409 | 2'061 | 669 | 176 | 977 | | |
| 4 | 145'904 | 161'732 | 79'458 | 46'642 | 29'384 | 15'811 | 3'598 | 5'527 | -2'484 | 462 | -1'018 | | | |
| 5 | 170'333 | 171'168 | 92'601 | 36'227 | 11'872 | 18'760 | 3'180 | 3'538 | 948 | -875 | | | | |
| 6 | 189'643 | 171'480 | 85'734 | 61'226 | 18'479 | 13'556 | 7'523 | 1'964 | 88 | | | | | |
| 7 | 179'022 | 217'202 | 101'080 | 56'183 | 28'362 | 29'791 | 11'244 | 12'568 | | | | | | |
| 8 | 205'908 | 210'139 | 104'397 | 45'277 | 34'888 | 30'193 | 17'563 | | | | | | | |
| 9 | 210'951 | 215'478 | 98'618 | 62'846 | 52'435 | 22'824 | | | | | | | | |
| 10 | 213'426 | 295'796 | 140'211 | 82'259 | 59'209 | | | | | | | | | |
| 11 | 249'508 | 330'502 | 142'126 | 122'023 | | | | | | | | | | |
| 12 | 258'425 | 427'587 | 229'097 | | | | | | | | | | | |
| 13 | 368'762 | 540'304 | | | | | | | | | | | | |
| 14 | 394'997 | | | | | | | | | | | | | |

# Case study from Merz-W.-Hashorva [1]

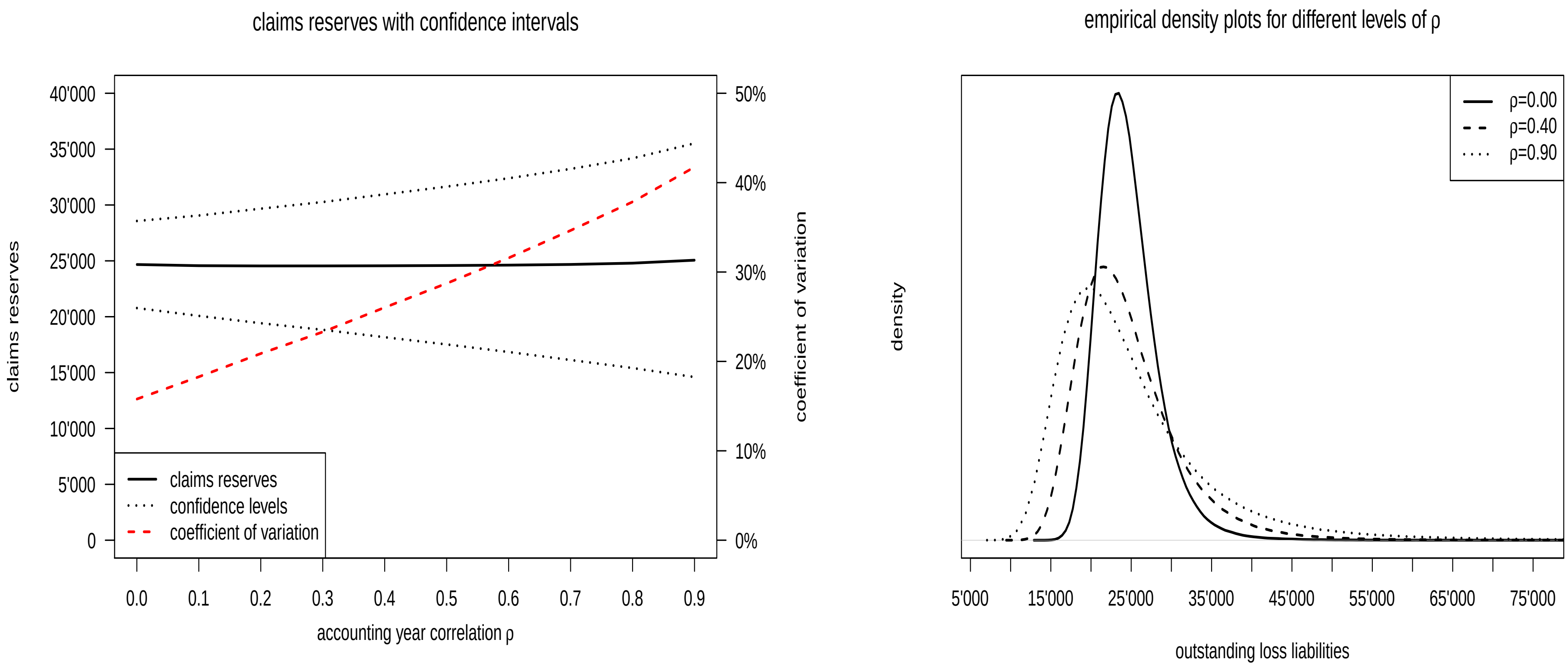



(a) lhs: prediction and 1-std.dev. confidence interval of total uncertainty and one-year CDR uncertainty as a function of $\rho \in [0, 1)$ for combined correlation (3).

(b) rhs: comparison (1) point-wise cells correlation, (2) calendar year correlation and (3) combined correlation for one-year CDR uncertainty and $\rho \in [0, 1)$.

# Case study from Salzmann-W. [2]




(a) lhs: prediction and 1-std.dev. confidence interval of total uncertainty as a function of calendar year correlation $\rho \in [0, 1)$.

(b) rhs: posterior density of outstanding loss liabilities for different calendar year correlation choices $\rho \in \{0.0,\ 0.4,\ 0.9\}$.

# Conclusions

- Dependencies and correlation have a huge effect on confidence intervals and quantiles: more case studies are needed!

- Effects of correlations are often counter-intuitive, see Conclusions 1-5 in Merz-W.-Hashorva [1].

- Our model allows in **solvency models** for a

  **bottom-up calibration of correlation**.

- Choices of reasonable correlation structures need to be investigated: Shi-Basu-Meyers [3] and W. [4] give some advice.

# References

[1] Merz, M., Wüthrich, M.V., Hashorva, E. (2012). Dependence modelling in multivariate claims run-off triangles. To appear in *Annals Actuarial Science*.

[2] Salzmann, R., Wüthrich, M.V. (2012). Modeling accounting year dependence in runoff triangles. To appear in *European Actuarial Journal*.

[3] Shi, P., Basu, S., Meyers, G.G. (2012). A Bayesian log-normal model for multivariate loss reserving. *North American Actuarial Journal* 16/1, 29-51.

[4] Wüthrich, M.V. (2012). Discussion of "A Bayesian log-normal model for multivariate loss reserving" by Shi-Basu-Meyers. To appear in *North American Actuarial Journal*.